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Partition numbers

Otmar Spinas[a,b,*]

[a] *Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, 10099 Berlin, Germany*

[b] *Mathematik, ETH-Zentrum, 8092 Zürich, Switzerland*

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Abstract

We continue [21] and study partition numbers of partial orderings which are related to $\mathscr{P}(\omega)/fin$. In particular, we investigate P_{f}, be the suborder of $(\mathscr{P}(\omega)/fin)^{\omega}$ containing only filtered elements, the Mathias partial order $\mathbb{M}$, and (ω), $(\omega)^{\omega}$ the lattice of (infinite) partitions of ω, respectively. We show that Solomon's inequality holds for $\mathbb{M}$ and that it consistently fails for P_{f}. We show that the partition number of (ω) is $\mathfrak{c}$. We also show that consistently the distributivity number of $(\omega)^{\omega}$ is smaller than the distributivity number of $\mathscr{P}(\omega)/fin$. We also investigate partitions of a Polish space into closed sets. We show that such a partition either is countable or has size at least $\mathfrak{d}$, where $\mathfrak{d}$ is the dominating number. We also show that the existence of a dominating family of size $\aleph_1$ does not imply that a Polish space can be partitioned into $\aleph_1$ many closed sets.

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0. Introduction and terminology

Using the terminology for Boolean algebras, by a *partition* of a partial order $(P, \leqslant)$ we understand a maximal family $\mathscr{A} \subseteq P$ of pairwise incompatible elements, i.e., for distinct $p,q \in \mathscr{A}$ no member of P is below p and q. In this case we write $p \perp q$. If p and q are compatible, i.e. there is $r \in P$ with $r \leqslant p,q$, we write $p \| q$. The set of elements of $\mathscr{A}$ which are compatible with p will be denoted by $\mathscr{A} \restriction p$. If P is given as a definition rather than a set, a partition $\mathscr{A}$ of P need not be absolute, e.g., it can be killed by some forcing which preserves cardinals. Typically, this is the case if $\mathscr{A}$

* E-mail: spinas@math.ethz.ch.

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is large enough, say infinite or uncountable. Then the partition number of P, denoted by $\mathfrak{a}(P)$, is the least cardinal such that there is no absolute partition of that size. In [21], $\mathfrak{a}(n)$ was defined as the minimal size of an infinite partition of $(\mathscr{P}(\omega)/fin)^n$, and, since $(\mathscr{P}(\omega)/fin)^\omega$ has absolute partitions which are countably infinite, $\mathfrak{a}(\omega)$ was defined as the minimal size of an uncountable partition of $(\mathscr{P}(\omega)/fin)^\omega$. In [21] it was proved that $\mathfrak{a}(N)\geqslant\mathfrak{b}$ holds for all $1\leqslant N\leqslant\omega$, thus generalizing Solomon's inequality $\mathfrak{b}\leqslant\mathfrak{a}$ (see [20] or [8, 3.1(a)]). Here $\mathfrak{b}$ is the minimal size of an unbounded family in $({}^\omega\omega,\leqslant^*)$, where $\leqslant^*$ is eventual dominance, and $\mathfrak{a}=\mathfrak{a}(1)$. It is well-known that $\mathfrak{b}$ is not absolute. The dominating number $\mathfrak{d}$ is defined as the minimal size of a cofinal set in $({}^\omega\omega,\leqslant^*)$. Clearly $\mathfrak{b}\leqslant\mathfrak{d}$.

Here we continue the investigation of partition numbers of partial orders related to $\mathscr{P}(\omega)/fin$. Let $P_f\subseteq(\mathscr{P}(\omega)/fin)^\omega$ be the subordering of filtered elements, and let $P_c\subseteq(\mathscr{P}(\omega)/fin)^\omega$ be the subordering of elements which are chains. This means, $p\in P_f$, P_c if and only if $p:\omega\to\mathscr{P}(\omega)/fin\setminus\{0\}$ and $ran(p)$ generates a filter, p is a descending chain on $P(\omega)/fin$, respectively. The order is coordinatewise. By $\mathscr{P}(\omega)/fin$ we will always mean $\mathscr{P}(\omega)/fin\setminus\{0\}$ in the sequel, and we confuse elements of $\mathscr{P}(\omega)/fin$ with their representatives in $[\omega]^\omega$. It is rather easy to see that P_f and P_c have absolute countably infinite partitions (see Lemma 1.3 below).

Hence, we define $\mathfrak{a}(P_f),\mathfrak{a}(P_c)$ as the minimal size of an uncountable partition of P_f, P_c, respectively. It is easy to see that $\mathfrak{a}(P_f)=\mathfrak{a}(P_c)$ (see Corollary 1.5). This is the right definition according to our philosophy; for we will show that $\mathfrak{p}\leqslant\mathfrak{a}(P_f)$. Here $\mathfrak{p}$ is the minimal cardinality of a filter on $\mathscr{P}(\omega)/fin$ which does not have a lower bound. It is well-known that $\mathfrak{p}$ is not absolute. We shall need an equivalent definition. Call a family $\mathscr{F}\subseteq[\omega]^\omega$ *trivially predense* if the union of some finite subset of $\mathscr{F}$ almost contains ω. Then $\mathfrak{p}$ is the minimal cardinality of a family $\mathscr{F}\subseteq[\omega]^\omega$ which is predense in the order $([\omega]^\omega,\subseteq^*)$ but not trivially predense. A closely related cardinal invariant is the tower number $\mathfrak{t}$. A well-ordered descending chain in $\mathscr{P}(\omega)/fin$ which does not have a lower bound is called a *tower*. Then $\mathfrak{t}$ is defined as the minimal length of a tower. Clearly $\mathfrak{p}\leqslant\mathfrak{t}$, but the consistency of $\mathfrak{p}<\mathfrak{t}$ is open. The number $\mathfrak{p}$ is smaller or equal than most other cardinal invariants, whereas partition numbers tend to be large. It is therefore surprising that one can show that $\mathfrak{p}$ is a sharp lower bound for $\mathfrak{a}(P_f)$ in the sense that $\mathfrak{p}=\omega_1$ implies $\mathfrak{a}(P_f)=\omega_1$. This result implies that consistently Solomon's inequality fails for P_f. In [22] the consistency of $\mathfrak{p}<\mathfrak{a}(P_f)$ with *ZFC* is proved.

The proof of $\mathfrak{p}\leqslant\mathfrak{a}(P_f)$ gives a rough classification of partitions of P_f into two classes. If a partition belongs to the first class it can be killed by adding a Cohen real. In other words, in this case the partition has size at least $cov(\mathscr{M})$, where $\mathscr{M}$ is the ideal of meagre subsets of the real line and $cov(\mathscr{M})$ is the minimal number of meagre sets which are needed to cover $\mathbb{R}$. It is well-known that $\mathfrak{p}\leqslant cov(\mathscr{M})$ holds and that $\mathfrak{p}<cov(\mathscr{M})$ is consistent. The latter is true in the Cohen model. If the partition belongs to the second class our forcing which kills it also increases $\mathfrak{p}$.

Let X be an uncountable Polish space. Let $\mathfrak{a}(X)$ be the minimal size of an uncountable partition of X into closed sets. By arguments from [16] one can show that $\mathfrak{a}(X)$ does not depend on X, so call it $\mathfrak{a}(cl)$. Stern [23] has proved that $cov(\mathscr{M})\leqslant\mathfrak{a}(cl)$. In

Section 2 we improve this by showing $\mathfrak{d} \leqslant \mathfrak{a}(cl)$. It is well-known that $cov(\mathcal{M}) \leqslant \mathfrak{d}$ and that $cov(\mathcal{M}) < \mathfrak{d}$ is consistent. Stern [23] and independently Baumgartner and Kunen have shown that $\mathfrak{a}(cl) < \mathfrak{c}$ is consistent. We also prove the consistency of $\mathfrak{d} < \mathfrak{a}(cl)$ with *ZFC*, by proving that Miller's forcing in [16] to kill a given partition of ${}^{\omega}2$ into closed sets is ${}^{\omega}\omega$-bounding. This answers negatively a question of Miller who asked whether the existence of a dominating family of size $\aleph_1$ implies that a Polish space can be partitioned into $\aleph_1$ many closed sets. By combining the proof with well-known results we even obtain the consistency of $cof(\mathcal{N}) < \mathfrak{a}(cl)$, where $cof(\mathcal{N})$ is the cofinality of the ideal $\mathcal{N}$ of Lebesgue measure zero sets of reals. By results about Cichoń's diagram (see [1]), $\mathfrak{d} \leqslant cof(\mathcal{N})$ holds in *ZFC* and $\mathfrak{d} < cof(\mathcal{N})$ is consistent.

In Section 3 we show that Solomon's inequality holds for the Mathias partial order.

In Section 4 we investigate partitions of ω, i.e. partitions of the partial order $(\mathcal{P}(\omega)\setminus\{\emptyset\}, \subseteq)$. Denote the set of all such (infinite) partitions by (ω), $(\omega)^{\omega}$, respectively. The relation of coarsening is a partial order on (ω). Krawczyk proved that the partition number of $(\omega)^{\omega}$ equals $\mathfrak{c}$. Halbeisen [9] and Krawczyk asked whether the same is true for the partition number of (ω). We give a positive answer. Let $\mathfrak{h}((\omega)^{\omega})$ denote the distributivity number of $(\omega)^{\omega}$, that is the minimal size of a family of partitions of $(\omega)^{\omega}$ which does not have a refinement. In [7] it was shown that $\mathfrak{h}((\omega)^{\omega}) \leqslant \mathfrak{h}$, where $\mathfrak{h}$ is the distributivity number of $\mathcal{P}(\omega)/fin$, and hence the question was raised of whether $\mathfrak{h}((\omega)^{\omega}) < \mathfrak{h}$ is consistent. We give a positive answer by showing that this is true in the Mathias model.

1. The partition number of the filter suborder of $(\mathcal{P}(\omega)/fin)^{\omega}$

Definition 1.1. (1) Let P_{f} be the subordering of $(\mathcal{P}(\omega)/fin)^{\omega}$ containing only filtered elements, i.e.,

$$P_{\mathrm{f}} = \{p \in (\mathcal{P}(\omega)/fin)^{\omega} : ran(p) \text{ generates a filter on } \mathcal{P}(\omega)/fin\}.$$

(2) Let P_{c} be the subordering of $(\mathcal{P}(\omega)/fin)^{\omega}$ containing only descending chains, i.e.,

$$P_{\mathrm{c}} = \{p \in (\mathcal{P}(\omega)/fin)^{\omega} : p \text{ is a descending chain in } \mathcal{P}(\omega)/fin\}.$$

(3) For every $p \in P_{\mathrm{f}}$ let $\bar{p} \in P_{\mathrm{c}}$ be defined by $\bar{p}(n) = \bigwedge_{i \leqslant n} p(i)$.

(4) For $A \in \mathcal{P}(\omega)/fin$, let $C_A \in P_{\mathrm{c}}$ be defined by $C_A(n) = A$ for all $n < \omega$.

Lemma 1.2. (1) *Suppose $p, q \in P_{\mathrm{f}}$ and p and q are compatible. Then p and $\bar{q}$ are compatible.*

(2) *Suppose that $\mathcal{A}$ is a partition of P_{f}. Then $\bar{\mathcal{A}} = \{\bar{p} : p \in \mathcal{A}\}$ is a partition of P_{f} and of P_{c}.*

(3) *Suppose that $\mathcal{A}$ is a partition of P_{c}. Then $\mathcal{A}$ is a partition of P_{f}.*

Proof. For (1), let $r \in P_f$ be such that $r \leqslant p, q$. It is easily seen that $\bar{r} \leqslant \bar{q}$. Hence $\bar{r} \leqslant p, \bar{q}$.

For (2), since $\bar{p} \leqslant p$ for $p \in P_f$ and $P_c \subseteq P_f$, it is clear that $\bar{\mathscr{A}}$ is pairwise incompatible. Let $p \in P_f$. By assumption there exists $q \in \mathscr{A}$ such that p and q are compatible. By the first part of the Lemma, p and $\bar{q}$ are compatible. But $\bar{q} \in \bar{\mathscr{A}}$. We conclude that $\bar{\mathscr{A}}$ is a partition of P_f and hence also of P_c.

(3) is clear. $\square$

Lemma 1.3. *The orderings P_f and P_c have partitions of size $\aleph_0$.*

Proof. By Lemma 1.2(3) it is enough to construct a countable partition of P_c. Let $(A_n : n \in \omega)$ be a partition of ω such that each A_n is infinite. For $n \in \omega$ let $p_n = C_{A_n}$. Let $p_\omega \in P_c$ be defined by $p_\omega(n) = \omega \setminus \bigcup\{A_i : i < n\}$. It is straightforward to check that $\{p_n : n \leqslant \omega\}$ is a partition of P_c. $\square$

Definition 1.4. Define $\mathfrak{a}(P_f), \mathfrak{a}(P_c)$ to be the minimal size of an uncountable partition of P_f, P_c, respectively.

Corollary 1.5. $\mathfrak{a}(P_f) = \mathfrak{a}(P_c)$.

Proof. Follows immediately from Lemma 1.2(2) and (3). $\square$

We will now show that it is consistent that $\mathfrak{a}(P_f)$ is much smaller than $\mathfrak{a}(\omega)$, the partition number of $(\mathscr{P}(\omega)/fin)^\omega$. This will follow from the following theorem.

Theorem 1.6. *If $\mathfrak{p} = \omega_1$, then $\mathfrak{a}(P_f) = \omega_1$.*

Proof. Rothberger proved that $\mathfrak{p} = \omega_1$ implies $\mathfrak{t} = \omega_1$ (see [17] or [8]). So let $(A_\alpha : \alpha < \omega_1)$ be a tower. We may assume that $\omega \setminus A_0$ and $A_\alpha \setminus A_{\alpha+1}$ are infinite for all $\alpha < \omega_1$. We will define $(p_\alpha : \alpha < \omega_1)$, a partition of P_f, by induction. Let $p_0 = C_{\omega \setminus A_0}$. For $n < \omega$, $n > 1$ let $p_n = C_{A_{n-1} \setminus A_n}$.

Suppose now that $\gamma < \omega_1$ is a limit and $(p_\alpha : \alpha < \gamma)$ has been defined. Choose an increasing sequence $(\alpha_n : n < \omega)$ with limit γ. Define p_γ by letting $p_\gamma(n) = A_{\alpha_n} \setminus A_\gamma$ for all $n \in \omega$, and let $p_{\gamma+n} = C_{A_{\gamma+n-1} \setminus A_{\gamma+n}}$ for all $n \in \omega \setminus 1$.

We claim that $(p_\alpha : \alpha < \omega_1)$ is a partition of P_f. First let us check incompatibility. Let $\alpha < \beta < \omega_1$. If β is a limit, then $p_\beta(n) \subseteq A_{\alpha+1}$ for almost all n, but $p_\alpha(n) \cap A_\alpha = \emptyset$ for all n. Since $A_{\alpha+1} \subseteq^* A_\alpha$, we conclude that p_α and p_β are incompatible.

If β is a successor, then $p_\beta \leqslant C_{A_{\beta-1}}$, but $p_\alpha \perp C_{A_{\beta-1}}$.

In order to prove maximality, note that it is enough to prove that every constant member of P_f is compatible with some p_α. Let $A \in [\omega]^\omega$ be arbitrary. Since $(A_\alpha : \alpha < \omega_1)$ is a tower we may choose $\gamma < \omega_1$ minimal such that $A \not\subseteq^* A_\gamma$, i.e., $A \setminus A_\gamma$ is infinite.

If γ is a limit, then C_A and p_γ are compatible, since $A \setminus A_\gamma \subseteq^* A_{\alpha_n}$ for all $n < \omega$, where $(\alpha_n : n < \omega)$ is as chosen in the definition of p_γ. Hence $C_{A \setminus A_\gamma} \leqslant C_A, p_\gamma$.

If γ is a successor, then C_A and p_γ are compatible, since $A\backslash A_\gamma \subseteq^* A_{\gamma-1}\backslash A_\gamma$, and hence $C_{A\backslash A_\gamma} \leqslant C_{A,p_\gamma}$.

Finally, if $\gamma = 0$ then clearly $C_{A\backslash A_0} \leqslant C_{A,p_0}$. $\square$

Corollary 1.7. *It is consistent with ZFC, relative to the consistency of ZFC itself, that* $\mathfrak{a}(P_{\mathrm{f}}) < \mathfrak{b}$.

Proof. Let V be a model for $ZFC + CH$. In V let $\kappa > \omega_1$ be a regular cardinal and let $(P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leqslant \kappa, \beta < \kappa)$ be a finite support iteration of Hechler forcing, i.e., the natural *c.c.c.* forcing to add a dominating function (see [3] for its definition). Let G be P_κ-generic over V. In [3] it was shown that $V[G]$ satisfies $\mathfrak{p} = \omega_1$. But $\mathfrak{b} = \kappa$ holds in $V[G]$ by standard arguments from [2]. $\square$

Corollary 1.8. *If ZFC is consistent, then so is* $ZFC + \mathfrak{a}(P_{\mathrm{f}}) < \mathfrak{a}(\omega)$.

Proof. By [21, Theorem 2.2], $\mathfrak{b} \leqslant \mathfrak{a}(\omega)$ holds. Hence the model for Corollary 1.7 can be used. $\square$

Next we will show that $\mathfrak{p} \leqslant \mathfrak{a}(P_{\mathrm{f}})$ holds. By Bell's Theorem (see [4]) this is essentially the same as proving that Martin's axiom for σ-centered posets implies $\mathfrak{a}(P_{\mathrm{f}}) = \mathfrak{c}$. Surprisingly, as in the case of $\mathfrak{a}(\omega)$, there does not seem to exist a trivial proof of this. As for $\mathfrak{a}(\omega)$ our proof gives a rough classification of partitions of P_{f}, so that according to which class a partition belongs one needs a different forcing to kill it. The first class is handled by the following lemma.

Lemma 1.9. *Let $\mathscr{A}$ be an uncountable partition of P_{c}. Suppose that for every $A \in \mathscr{P}(\omega)/fin$, if $\mathscr{A} \restriction C_A$ is uncountable, then there exists $p \in \mathscr{A} \restriction C_A$ such that $\mathscr{A} \restriction C_{A \cap (p(n)\backslash p(n+1))}$ is uncountable for infinitely many $n < \omega$. Then $\mathscr{A}$ has size at least cov$(\mathscr{M})$.*

Proof. To simplify our notation, we will often write A instead of C_A. Fix a function $g : \omega \to \omega$ such that

(i) $g(n) < n$ for all $n \in \omega\backslash\{0\}$,

(ii) $g^{-1}\{n\}$ is infinite for all $n < \omega$.

By induction we will construct sequences $(p_n : n \in \omega), (q_n : n \in \omega)$ and $(k_n : n \in \omega)$ such that the following requirements are satisfied:

(1) $p_n \in \mathscr{A}$ and $q_n \in P_{\mathrm{c}}$ and there exist infinitely many k such that $\mathscr{A} \restriction q_n(k)\backslash q_n(k+1)$ is uncountable;

(2) $(k_n : n < \omega)$ is an increasing sequence of natural numbers such that $p_i(k_n) \cap p_j(k_n) =^* \emptyset$ for all distinct $i, j \leqslant n$, and moreover $\mathscr{A} \restriction q_{g(n+1)}(k_n)\backslash q_{g(n+1)}(k_n+1)$ is uncountable;

(3) if $n > 0$, then $q_n(i) = p_n(i) \cap (q_{g(n)}(k_{n-1})\backslash q_{g(n)}(k_{n-1} + 1))$ for all $i \geqslant k_{n-1}$, and $q_n(i) = q_{g(n)}(i)$ for all $i < k_{n-1}$.

For the construction, for $n=0$ choose $p_0 \in \mathscr{A}$ such that there exist infinitely many k such that $\mathscr{A} \restriction p_0(k) \setminus p_0(k+1)$ is uncountable. This is possible by assumption on $\mathscr{A}$. Let $q_0 = p_0$ and choose k_0 such that $\mathscr{A} \restriction p_0(k_0) \setminus p_0(k_0+1)$ is uncountable. Then (1) and (2) hold for $n=0$. Now suppose we are at stage $n>0$. By induction hypothesis (2) we know that $\mathscr{A} \restriction q_{g(n)}(k_{n-1}) \setminus q_{g(n)}(k_{n-1}+1)$ is uncountable. Define $A = q_{g(n)}(k_{n-1}) \setminus q_{g(n)}(k_{n-1}+1)$. By assumption on $\mathscr{A}$ we may find $p_n \in \mathscr{A} \restriction A$ such that $\mathscr{A} \restriction (p_n(k) \setminus p_n(k+1)) \cap A$ is uncountable for infinitely many $k<\omega$. Note that by induction hypothesis (2) and (3), $p_n \neq p_i$ for all $i<n$. Indeed, by (3) we have that if $g(n)>0$, then

$$q_{g(n)}(i) = p_{g(n)}(i) \cap (q_{g^2(n)}(k_{g(n)-1}) \setminus q_{g^2(n)}(k_{g(n)-1}+1)) \tag{$*$}$$

for all $i \geqslant k_{g(n)-1}$, and thus, by (2), for every $i \geqslant k_{n-1}$. Since $p_n \in \mathscr{A} \restriction A$, by (2) we conclude that $p_n \neq p_i$ for all $i \in n \setminus \{g(n)\}$. By $(*)$ we have that $A \subseteq p_{g(n)}(k_{n-1}) \setminus p_{g(n)}(k_{n-1}+1)$. Hence we also have $p_n \neq p_{g(n)}$. Now suppose $g(n)=0$. Then $A = p_0(k_{n-1}) \setminus p_0(k_{n-1}+1)$. Hence $p_n \neq p_0$, and by (2) also $p_n \neq p_i$ for all $i \in n \setminus \{0\}$.

Now define q_n so that (3) for n becomes true. Then clearly (1) holds, since $(p_n(k) \setminus p_n(k+1)) \cap A = q_n(k) \setminus q_n(k+1)$ for almost all k.

Finally, we choose $k_n > k_{n-1}$ such that (2) holds. If $g(n+1)=n$ this is possible, since, as we just proved, q_n satisfies (1), and all p_i, $i \leqslant n$, are distinct, hence incompatible and therefore for almost all k, $p_i(k) \cap p_j(k) =^* \emptyset$ whenever $i, j \leqslant n$ are distinct. If $g(n+1)<n$ we apply induction hypothesis (1) for $q_{g(n+1)}$. It is now easy to check that (1)–(3) are satisfied for n. This finishes the construction.

Let $T = \{q_n \restriction k : k, n \in \omega\}$. It is immediate from the construction and property (ii) of g that $(T, \subseteq)$ is a perfect tree, and that every branch of T belongs to P_c. We endow the set of branches of T, denoted with $[T]$, with the usual topology. This means that basic open sets are $[t] := \{x \in [T] : t \subset x\}$, where $t \in T$. Then the following claim suffices to finish the proof of Lemma 1.9.

Claim. *Letting $C_p = \{q \in [T] : p$ is compatible with $q\}$, C_p is nowhere dense in $[T]$, for every $p \in \mathscr{A}$.*

Proof of the Claim. Let $p \in \mathscr{A}$. First note that C_p is closed. In fact, given q in the closure of C_p, for every k there is $q' \in C_p$ with $q' \restriction k+1 = q \restriction k+1$. Hence, $p(k) \cap q(k)$ is infinite for every k, and so $q \in C_p$. Second, $\{q_n : n \in \omega\}$ is dense in $[T]$ by construction. Hence, if C_p were somewhere dense in $[T]$, there exists $t \in T$ such that $[t] \subseteq C_p$. Then C_p contains infinitely many of the q_n. Note that $q_n \in C_p$ implies $p_n \in C_p$. Indeed, let $r \in P_c$ such that $r \leqslant p, q_n$, and define $r' \in P_c$ as follows:

$$r'(i) = \begin{cases} r(i) & \text{if } i \geqslant k_{n-1}, \\ q_n(k_{n-1}) \cap r(i) & \text{if } i < k_{n-1}. \end{cases}$$

Then $r' \leqslant r$, and from (3) we conclude $r' \leqslant p_n$. Consequently, p is compatible with infinitely many of the p_n. Since the p_n are all distinct, this is a contradiction. □

If the hypothesis of Lemma 1.9 fails, then the situation clears up, and a partition of P_f can be killed in a similar way as this is done with partitions of $\mathscr{P}(\omega)/fin$.

Lemma 1.10. *Let $\mathscr{A}$ be a partition of P_c. Suppose that there exists $A \in \mathscr{P}(\omega)/fin$ such that $\mathscr{A} \upharpoonright C_A$ is uncountable, but for every $p \in \mathscr{A} \upharpoonright C_A$ there exist only finitely many n such that $\mathscr{A} \upharpoonright C_{A\cap(p(n)\setminus p(n+1))}$ is uncountable. Then $\mathscr{A}$ has size at least $\mathfrak{p}$.*

Proof. For $p \in \mathscr{A}$ fix $n_p \in \omega$ such that $\mathscr{A} \upharpoonright C_{A\cap(p(n)\setminus p(n+1))}$ is countable for all $n \geqslant n_p$.

Claim. *$\mathscr{A} \upharpoonright C_{A\cap p(n_p)}$ is countable.*

Proof of the Claim. Just note that $\{p\} \cup \bigcup\{\mathscr{A} \upharpoonright C_{A\cap(p(n)\setminus p(n+1))}: n \geqslant n_p\}$ induces a partition of $C_{A\cap p(n_p)}$. For, given $B \in \mathscr{P}(\omega)/fin$ with $C_B \leqslant C_{A\cap p(n_p)}$, if $B \cap p(n)\setminus p(n+1)$ is infinite for some $n \geqslant n_p$ then C_B is compatible with a member of $\mathscr{A} \upharpoonright C_{A\cap(p(n)\setminus p(n+1))}$. Otherwise $B \subseteq^* p(n)$ for all n, and hence $C_B \leqslant p$. □

By the Claim it is clear that $\{p(n_p) : p \in \mathscr{A} \upharpoonright C_A\}$ is not trivially predense on A. Indeed, if there were $p_0, \ldots, p_{n-1} \in \mathscr{A} \upharpoonright C_A$ such that $A \subseteq^* \bigcup_{i<n} p_i(n_{p_i})$, then every $p \in P_c$ with $p \leqslant C_A$ is compatible with some $C_{A\cap p_i(n_{p_i})}$. Hence $\mathscr{A} \upharpoonright C_A = \bigcup_{i<n} \mathscr{A} \upharpoonright C_{A\cap p_i(n_{p_i})}$ is countable, a contradiction.

Hence, if $|\mathscr{A}| < \mathfrak{p}$, by using the dual definition of $\mathfrak{p}$ mentioned in the introduction we can find $B \in \mathscr{P}(\omega)/fin$ with $B \leqslant A$ such that $B \cap p(n_p) =^* \emptyset$ for all $p \in \mathscr{A} \upharpoonright C_A$. But then no member of $\mathscr{A}$ is compatible with C_B. This contradicts the maximality of $\mathscr{A}$. □

Putting together Lemmas 1.9 and 1.10 we obtain the following:

Theorem 1.11. $\mathfrak{a}(P_f) \geqslant \mathfrak{p}$.

Questions. Is $\mathfrak{a}(P_f) = \mathfrak{p}$ true?[1] Is $\mathfrak{t} \leqslant \mathfrak{a}(P_f)$ true?

2. Partitioning a Polish space into closed sets

Definition 2.1. Let X be an uncountable Polish space. Define $\mathfrak{a}(X)$ as the minimal size of an uncountable partition of X into closed sets.

The following lemma is essentially [16, Theorem 3]. Therefore we only sketch the proof.

Lemma 2.2. *For any uncountable Polish spaces X, Y we have $\mathfrak{a}(X) = \mathfrak{a}(Y)$.*

Proof. Since every Polish space is the continuous image of ${}^{\omega}\omega$ we have $\mathfrak{a}(X) \geqslant \mathfrak{a}({}^{\omega}\omega)$. Let us show $\mathfrak{a}({}^{\omega}\omega) \geqslant \mathfrak{a}({}^{\omega}2)$. Let $\mathscr{C} = \{\mathscr{C}_\alpha : \alpha < \kappa\}$ be a partition of ${}^{\omega}\omega$ into closed sets, where $\kappa \geqslant \omega_1$. Build $P \subseteq {}^{\omega}\omega$ compact and perfect so that for some countable $\mathscr{Y} \subseteq \mathscr{C}, \bigcup \mathscr{Y}$ is dense in P but every $C \in \mathscr{Y}$ is nowhere dense in P. By the Baire Category Theorem

[1] This is answered negatively in [22].

it follows that $\mathscr{C}$ induces an uncountable partition of P. Since P is homeomorphic to ${}^{\omega}2$, we conclude $\mathfrak{a}({}^{\omega}\omega) \geqslant \mathfrak{a}({}^{\omega}2)$.

Next we show that $\mathfrak{a}({}^{\omega}2) \geqslant \mathfrak{a}([0,1])$. Let $(C_\alpha : \alpha < \kappa)$ be an uncountable partition of ${}^{\omega}2$ into closed sets. We may assume that every C_α contains at most one real which is eventually constant. Define $F : {}^{\omega}2 \to [0,1]$ by

$$F(x) = \sum_{n<\omega} \frac{x(n)}{2^{n+1}},$$

and let $D_\alpha = F[C_\alpha]$. Since for $x \neq y$, $F(x) = F(y)$ implies that x and y are finally constant, we conclude that for every α there is at most one $\beta \neq \alpha$ with $D_\alpha \cap D_\beta \neq \emptyset$. By glueing any two such D_α, D_β, we obtain a partition of [0,1] into closed sets which has size κ.

Finally let X be arbitrary. Embed X into $[0,1]^\omega$ (see [11, Theorem 4.14]). If some projection of X contains an interval, then by pulling back a partition of that interval we obtain $\mathfrak{a}(X) \leqslant \mathfrak{a}([0,1])$; hence $\mathfrak{a}(X) = \mathfrak{a}({}^{\omega}\omega)$ by what we have shown above. Otherwise X is zero-dimensional. Thus either X contains a clopen set homeomorphic to ${}^{\omega}2$ or X is homeomorphic to ${}^{\omega}\omega$ (see [11, Theorem 7.8]), and we obtain $\mathfrak{a}(X) = \mathfrak{a}({}^{\omega}2)$ or $\mathfrak{a}(X) \leqslant \mathfrak{a}({}^{\omega}\omega)$, accordingly. But above we have shown $\mathfrak{a}({}^{\omega}\omega) = \mathfrak{a}({}^{\omega}2)$. Again we get $\mathfrak{a}(X) = \mathfrak{a}({}^{\omega}\omega)$. $\square$

Definition 2.3. Let $\mathfrak{a}(cl)$ denote $\mathfrak{a}(X)$ where X is any uncountable Polish space.

Lemma 2.4. *Let C be a countable set and let $(a_\alpha : \alpha < \kappa)$ be such that $a_\alpha \in [C]^\omega$ and $\kappa \geqslant \omega_1$.*

(1) *For only countably many $\alpha < \kappa$ can there be $F_\alpha \in [a_\alpha]^{<\omega}$ such that $\{\beta < \kappa : F_\alpha \subseteq a_\beta\}$ is countable.*

(2) *There exists $Y \in [\kappa]^\omega$ such that $\forall \alpha \in Y \, \forall F \in [a_\alpha]^{<\omega} \exists^\infty \beta \in Y (F \subseteq a_\beta)$.*

Proof. We leave the proof of (2) to the reader, as it easily follows from (1). Suppose (1) were false, hence there exists $X \in [\kappa]^{\omega_1}$, and for every $\alpha \in X$ there exists $F_\alpha \in [a_\alpha]^{<\omega}$ such that there are only countably many $\beta < \kappa$ with $F_\alpha \subseteq a_\beta$. It is easy to construct an increasing sequence $(\alpha_\nu : \nu < \omega_1)$ of ordinals in X such that $F_{\alpha_\mu} \nsubseteq a_{\alpha_\nu}$ for all $\nu < \omega_1$ and $\mu < \nu$. But then the F_{α_ν} are pairwise distinct, which is impossible. $\square$

Theorem 2.5. $\mathfrak{a}(cl) \geqslant \mathfrak{d}$.

Proof. Let $(C_\alpha : \alpha < \kappa)$ be an uncountable partition of ${}^{\omega}2$ into nonempty closed sets. Choose trees $p_\alpha \subseteq {}^{<\omega}2$ such that $C_\alpha = [p_\alpha]$. By Lemma 2.4 we get $A \in [\kappa]^\omega$ such that $\forall \alpha \in A \forall \sigma \in p_\alpha \exists^\infty \beta \in A (\sigma \in p_\beta)$. Note that then every p_α, $\alpha \in A$, is nowhere dense. By renumbering we arrange that $A = \omega$.

For $\sigma \in {}^{<\omega}2$ and $i < \omega$ let $j(\sigma, i)$ be the least $j > i$, $j < \omega$, with $\sigma \in p_j$, if there is such j. For every $\alpha \in \kappa \setminus \omega$ we define $f_\alpha \in {}^{\omega}\omega$ together with $(A_n^\alpha : n < \omega)$ as follows: Let $f_\alpha(0) = \max\{|\sigma| : \sigma \in p_0 \cap p_\alpha\}$, and let $A_0^\alpha = \{j(\sigma, 0) : \sigma \in p_0 \cap p_\alpha\}$. Then A_0^α is finite.

Suppose we have defined $f_\alpha(n)$ and finite set $A^\alpha_n \subseteq \omega \setminus (n+1)$. Let

$$f_\alpha(n+1) = \max\left\{ |\sigma| : \sigma \in \bigcup_{i \in A^\alpha_n} p_i \cap p_\alpha \right\},$$

$$A^\alpha_{n+1} = \{ j(\sigma, i) : i \in A^\alpha_n \wedge \sigma \in p_i \cap p_\alpha \}.$$

Note that A^α_{n+1} is finite.

Claim. $(f_\alpha : \omega \leqslant \alpha < \kappa)$ *is dominating.*

Proof. Let $g \in {}^\omega\omega$. Define $x \in {}^\omega 2$ as follows. Define $(\sigma_n : n < \omega)$ and $(i_n : n < \omega)$ and then let $x = \bigcup\{\sigma_n : n < \omega\}$. Let σ_0 be the lexicographically least member of $p_0 \cap {}^{g(0)}2$ and let $i_0 = 0$. Suppose we have gotten σ_n and i_n such that $\sigma_n \in p_{i_n}$. Let $i_{n+1} = j(\sigma_n, i_n)$. Choose $\sigma_{n+1} \in p_{i_{n+1}}$ lexicographically least such that $\sigma_n \subseteq \sigma_{n+1}$, $\sigma_{n+1} \notin p_{i_n}$, and $|\sigma_{n+1}| \geqslant g(n+1)$. This finishes the definition of x.

Choose $\alpha < \kappa$ such that $x \in C_\alpha$. Note that $\alpha \notin \omega$, since $\sigma_n \notin \bigcup_{j < i_n} p_j$ for all $n < \omega$.

By induction we show that $f_\alpha(n) \geqslant g(n)$. Since $\sigma_0 \in p_0 \cap p_\alpha$ and $|\sigma_0| = g(0)$ we have that $f_\alpha(0) \geqslant g(0)$ by definition of $f_\alpha(0)$. We have $i_1 = j(\sigma_0, 0)$ and hence $i_1 \in A^\alpha_0$. Suppose we have shown that $f_\alpha(n) \geqslant g(n)$ and $i_{n+1} \in A^\alpha_n$. By construction we have $\sigma_{n+1} \in p_{i_{n+1}} \cap p_\alpha$ and $|\sigma_{n+1}| \geqslant g(n+1)$. By definition we conclude $f_\alpha(n+1) \geqslant |\sigma_{n+1}| \geqslant g(n+1)$. $\square$

Remark. (a) Miller has found a topological proof of Theorem 2.5.

(b) The main theorem of [16] is the consistency of $\max\{cov(\mathcal{M}), \mathfrak{b}\} < \mathfrak{a}(cl)$. This is a corollary of Theorem 2.5 and the well-known fact that in a model obtained by forcing with a countable-support iteration of length ω_2 of rational perfect set forcing over a model of *CH* we have $cov(\mathcal{M}) = \mathfrak{b} = \omega_1$ and $\mathfrak{d} = \omega_2$.

In the sequel we shall prove the consistency of $\mathfrak{d} < \mathfrak{a}(cl)$ with *ZFC*. For this it will be enough to prove the Miller's forcing in [16] is ${}^\omega\omega$-bounding, i.e. every function in ${}^\omega\omega$ in the extension is bounded by some function in the ground model.

Definition 2.6. Given $\mathscr{C} = (C_\alpha : \alpha < \kappa)$ an uncountable partition of ${}^\omega 2$ into closed sets, let $Q(\mathscr{C})$ be the set of perfect trees p on ${}^{<\omega}2$ such that no C_α is somewhere dense in $[p]$. The order of $Q(\mathscr{C})$ is inclusion.

It is clear that forcing $Q(\mathscr{C})$ adds a real which does not belong to $\bigcup \mathscr{C}$. Miller [16] has shown that $Q(\mathscr{C})$ has the *Laver property*, i.e. given $\underset{\sim}{x}$ a name for a member of ${}^\omega\omega$ with the property that for some $p \in Q(\mathscr{C})$ and $h \in {}^\omega\omega$ in the ground model we have $p \Vdash \forall n(\underset{\sim}{x}(n) \leqslant h(n))$, there is $q \leqslant p$, $q \in Q(\mathscr{C})$, and $H : \omega \to [\omega]^{<\omega}$ in the ground model such that $|H(n)| \leqslant n+1$ and $q \Vdash \forall n(\underset{\sim}{x}(n) \in H(n))$. Below we shall show that $Q(\mathscr{C})$ even has the *Sacks property*, i.e. the conclusion above holds for every $\underset{\sim}{x}$, a $Q(\mathscr{C})$-name for a real. By Miller's result, it certainly suffices to show that $Q(\mathscr{C})$ is ${}^\omega\omega$-bounding. Given tree p and $\sigma \in p$, let $p(\sigma)$ be the subtree of those nodes of p which are comparable with σ.

Lemma 2.7. *Forcing $Q(\mathscr{C})$ is ${}^{\omega}\omega$-bounding.*

Proof. Let $\underset{\sim}{g}$ be a $Q(\mathscr{C})$-name for a member of ${}^{\omega}\omega$, and let $p \in Q(\mathscr{C})$. First we construct $q \leqslant p, q \in Q(\mathscr{C})$, and $X = \{x_\sigma : \sigma \in {}^{<\omega}\omega\}$ such that

(1) X is dense in $[q]$.

(2) If $i<j<\omega$ there is $m<\omega$ such that $x_{\sigma^\frown i} \restriction m \neq x_\sigma \restriction m$ and $x_{\sigma^\frown j} \restriction m = x_\sigma \restriction m$. Moreover, if $\rho \subseteq \sigma \subseteq \tau$ $\rho \neq \sigma \neq \tau$, then $x_\sigma \restriction m = x_\rho \restriction m$ and $x_\sigma(m) \neq x_\rho(m)$ implies that $x_\tau \restriction m = x_\rho \restriction m$ and $x_\tau(m) \neq x_\rho(m)$. Let $m(\sigma^\frown i)$ be the unique m with $x_{\sigma^\frown i} \restriction m = x_\sigma \restriction m$ and $x_{\sigma^\frown i}(m) \neq x_\sigma(m)$. Let $m(\emptyset) = 0$.

(3) If $x_{\sigma^\frown i} \restriction m \not\subseteq x_\sigma$ then $q(x_{\sigma^\frown i} \restriction m)$ decides $\underset{\sim}{g} \restriction |\sigma| + 1$, say as $\rho(\sigma^\frown i)$.

For the construction, let $x_\emptyset$ be the leftmost branch of p. Let $\alpha_\emptyset$ be such that $x_\emptyset \in C_{\alpha_\emptyset}$, and let $p_0 = p$. Suppose that for some $n<\omega$ we have gotten $p_n, x_\sigma, \alpha_\sigma$ with $x_\sigma \in C_{\alpha_\sigma} \cap [p_n]$, for every $\sigma \in {}^n\omega$. Fix such σ. Let $(m_i : i<\omega)$ increasingly enumerate all $m > m(\sigma)$ such that $x_\sigma \restriction m^\frown\langle 1 - x_\sigma(m)\rangle \in p_n$. For every i choose $p^i_\sigma \leqslant p_n(x_\sigma \restriction m_i{}^\frown\langle 1 - x_\sigma(m_i)\rangle)$ such that $[p^i_\sigma] \cap C_{\alpha_\sigma} = \emptyset$ and p^i_σ decides $\underset{\sim}{g} \restriction n+1$. Let $p_{n+1} = \bigcup\{p^i_\sigma : i<\omega, \sigma \in {}^n\omega\}$, let $x_{\sigma^\frown i}$ be the leftmost branch of p^i_σ, and let $x_{\sigma^\frown i} \in C_{\alpha_{\sigma^\frown i}}$. Finally let $q = \bigcap_{n<\omega} p_n$. Then $\{x_\sigma : \sigma \in {}^{<\omega}\omega\} \subseteq [q]$, and since x_σ and $x_{\sigma^\frown i}$ belong to different C_α's, it is clear that $q \in Q(\mathscr{C})$. It is easy to check that (1)–(3) hold.

By a similar fusion argument as we just did, it is easy to see that for every $r \in Q(\mathscr{C})$ and $A \in [\kappa]^\omega$ there is $r_1 \leqslant r$, $r_1 \in Q(\mathscr{C})$, such that $[r_1] \cap C_\alpha = \emptyset$, all $\alpha \in A$. In fact, in the above construction choose $x_\emptyset \in [p] \setminus \bigcup_{\alpha\in A} C_\alpha$ (by Baire Category Theorem), and for $\sigma \in {}^n\omega$, $n>0$, choose $p^i_\sigma \leqslant p_n(x_\sigma \restriction m_i{}^\frown\langle 1 - x_\sigma(m_i)\rangle)$ such that $[p^i_\sigma] \cap (C_{\alpha_\sigma} \cup C_{\alpha_n}) = \emptyset$, where $(\alpha_n : n<\omega)$ enumerates A, and then choose $x_{\sigma^\frown i} \in [p^i_\sigma] \setminus \bigcup_{\alpha\in A} C_\alpha$.

Hence we can find $q_1 \leqslant q$, $q_1 \in Q(\mathscr{C})$, such that $[q_1] \cap X = \emptyset$. Let

$$T = \{\emptyset\} \cup \{\sigma^\frown i : x_{\sigma^\frown i} \restriction m(\sigma^\frown i) + 1 \in q_1\}.$$

Note that T is a finitely branching tree, and that $\{x_\tau \restriction m(\tau) + 1 : \tau \in T \cap {}^n\omega\}$ is a front of q_1, for every $0<n<\omega$. Hence, if we define $h \in {}^\omega\omega$ by letting

$$h(n) = \max\{\rho(\sigma^\frown i)(|\sigma|) + 1 : \sigma^\frown i \in T \wedge |\sigma| = n\},$$

then by construction we have that $q_1 \Vdash \forall n(\underset{\sim}{g}(n) < h(n))$. $\square$

Theorem 2.8. *It is consistent, relative to the consistency of ZF, that $cof(\mathscr{N}) < \mathfrak{a}(cl)$.*

Proof. Let V be a model of $ZFC + GCH$ and let P be a countable support iteration of length ω_2 of forcings $Q(\mathscr{C})$, where we make sure that for every partition of ${}^\omega 2$ into $\aleph_1$ closed sets $\mathscr{C}$ which occurs during the iteration, $Q(\mathscr{C})$ is an interand. By a well-known result of Shelah [18], the Sacks property is preserved by countable support iterations. Hence, by Lemma 2.7 and Miller's result that $Q(\mathscr{C})$ has the Laver property, P has the Sacks property. By well-known arguments (see [1]) forcing with the Sacks property over a model of CH preserves $cof(\mathscr{N}) = \omega_1$. Hence in the extension of V by P we have $cof(\mathscr{N}) = \omega_1$ and $\mathfrak{a}(cl) = \omega_2$. $\square$

Remark. The consistency of $ZFC + \mathfrak{a}(cl) < cof(\mathcal{N})$ was proved independently by Kunen and Stern (see [16]). A model for this is the random real model.

3. The partition number of the Mathias partial order

Here we show that Solomon's inequality is true for the Mathias partial order.

Definition 3.1. The Mathias partial order $\mathbb{M}$ is defined as follows. Elements of $\mathbb{M}$ are pairs $(s,a) \in [\omega]^{<\omega} \times [\omega]^{\omega}$ such that $\max(s) < \min(a)$. The order on $\mathbb{M}$ is defined by letting $(s,a) \leqslant (t,b)$ if $t \subseteq s \subseteq t \cup b$ and $a \subseteq b$.

The partial order $\mathbb{M}$ was introduced in [15]. Note that $\mathbb{M}$ has countably infinite partitions. For example, $\{(\{n\}, \omega \backslash n+1) : n < \omega\}$ is one. Hence, we define the partition number of $\mathbb{M}$ as follows.

Definition 3.2. The partition number of $\mathbb{M}$, denoted by $\mathfrak{a}(\mathbb{M})$, is the minimal size of an uncountable partition of $\mathbb{M}$.

Theorem 3.3. $\mathfrak{a}(\mathbb{M}) \geqslant \mathfrak{b}$.

Proof. Let $((u_\alpha, a_\alpha) : \alpha < \kappa)$ be an uncountable partition of $\mathbb{M}$. Fix u such that for some uncountable $X \subseteq \kappa$ we have $u_\alpha = u$ for all $\alpha \in X$. By Lemma 2.4 we find $Y \in [X]^{\omega}$ with the property that for every $\alpha \in Y$ and $F \in [a_\alpha]^{<\omega}$ there exist infinitely many $\beta \in Y$ such that $F \subseteq a_\beta$. Let $(\alpha_n : n < \omega)$ be the increasing enumeration of Y.

Next, for each $\alpha < \kappa$ define $f_\alpha : Y \to \omega$ by letting

$$f_\alpha(\alpha_n) = \begin{cases} \max(a_\alpha \cap a_{\alpha_n}) & \text{if } |a_\alpha \cap a_{\alpha_n}| < \omega, \\ 0 & \text{otherwise.} \end{cases}$$

If $\kappa < \mathfrak{b}$ choose $g : Y \to \omega$ such that for all $\alpha < \kappa$ we have $g(\alpha_n) > f_\alpha(\alpha_n)$ for almost all $n < \omega$. Moreover, since the $a_\alpha, \alpha \in X$, are almost disjoint, we may additionally assume that $\bigcup_{i<n} a_{\alpha_i} \cap a_{\alpha_n} \subseteq g(\alpha_n)$, for all $n < \omega$.

Now define $a = \{l_k : k \in \omega\} \in [\omega]^{\omega}$ and an increasing sequence $(n_k : k < \omega)$ by induction such that for each $k < \omega$ the following requirements are satisfied:

(1) $l_k \in a_{\alpha_{n_k}} \setminus g(\alpha_{n_k})$,

(2) $\{l_i : i \leqslant k\} \subseteq a_{\alpha_{n_k}}$.

For the construction, for $k=0$ let $n_0 = 0$ and choose $l_0 \in a_{\alpha_0} \setminus g(\alpha_0)$ arbitrarily. At stage $k > 0$, by induction hypothesis (2) and the choice of Y, there exists $n_k > n_{k-1}$ such that $\{l_k : i < k\} \subseteq a_{\alpha_{n_k}}$. Choose $l_k \in a_{\alpha_{n_k}} \setminus g(\alpha_{n_k})$ arbitrarily. The following claim shows that our assumption $\kappa < \mathfrak{b}$ is false.

Claim. *(u,a) is incompatible with (u_α, a_α) for every $\alpha < \kappa$.*

Proof of the Claim. Fix $\alpha < \kappa$. We have three cases to consider.

Case 1: u_α is a proper initial segment of u and $u \setminus u_\alpha \subseteq a_\alpha$. In this case $a_\alpha \cap a_\beta$ is finite for all $\beta \in X$, hence in the definition of $f_\alpha(\alpha_n)$ the first case occurred for all n. It is therefore clear that $a \cap a_\alpha$ is finite.

Case 2: u is a proper initial segment of u_α and $u_\alpha \setminus u \subseteq a$. Choose k such that $u_\alpha \setminus u \subseteq \{l_i : i < k\}$. Then by property (2) of a, for every $k' \geqslant k$ we have $u_\alpha \setminus u \subseteq a_{n_{k'}}$ and hence $a_{n_{k'}} \cap a_\alpha$ must be finite. Therefore, in the definition of $f_\alpha(\alpha_{n_{k'}})$ the first case occurred for all $k' \geqslant k$, and so clearly $a \cap a_\alpha$ is finite.

Case 3: $u_\alpha = u$. If $\alpha \in Y$, then $a \cap a_\alpha$ is finite by the "moreover" requirement on g. If $\alpha \in X \setminus Y$, then $a_\alpha \cap a_{\alpha_n}$ is finite for all $n < \omega$, and hence in the definition of f_α, the first case occurred always. Hence $a \cap a_\alpha$ is finite, as $g >^* f_\alpha$. □

Fact 3.4. $\mathfrak{a}(\mathbb{M}) \leqslant \mathfrak{a}$.

Proof. Let $\mathscr{A}$ be an infinite maximal almost disjoint family on ω, i.e. an infinite partition of $\mathscr{P}(\omega)/fin$. Define a partition $\mathscr{B}$ of $\mathbb{M}$ as follows:

$$\mathscr{B} = \{(s, a \setminus (\max(s)+1)) : s \in [\omega]^{<\omega} \text{ and } a \in \mathscr{A}, \text{ and either } s = \emptyset \text{ or } \max(s) \notin a\}.$$

First let us check that $\mathscr{B}$ is pairwise incompatible. Let $p = (s, a \setminus (\max(s)+1))$ and $q = (t, b \setminus (\max(t)+1))$ be distinct members of $\mathscr{B}$. They are certainly incompatible if $a \neq b$. So let us assume $a = b$ but $s \neq t$. If neither $s \subset t$ nor $t \subset s$ holds, p, q are incompatible. Without loss of generality we may therefore assume that $s \subset t$. But then $\max(t) \notin a$ and hence p and q are incompatible. Second, to check maximality let $(s, a) \in \mathbb{M}$ be arbitrary. Choose $b \in \mathscr{A}$ such that $a \cap b$ is infinite. If $s \subseteq b$, then $(\emptyset, b)$ and (s, a) are compatible and $(\emptyset, b) \in \mathscr{B}$. If $s \nsubseteq b$ let t be the maximal initial segment of s with $\max(t) \notin b$. Then $(t, b) \in \mathscr{B}$ and (t, b) and (s, a) are compatible. □

Questions. (a) Is $\mathfrak{a}(\mathbb{M}) = \mathfrak{a}$ true?

(b) Let $\mathbb{L}$ denote Laver forcing (see [13]), and let $\mathfrak{a}(\mathbb{L})$ be the partition number of $\mathbb{L}$, that is, the minimal size of an uncountable partition of $\mathbb{L}$. Is $\mathfrak{b} \leqslant \mathfrak{a}(\mathbb{L})$ true? Is there any provable relation between $\mathfrak{a}(\mathbb{M})$ and $\mathfrak{a}(\mathbb{L})$?

4. The partition and distributivity numbers of the lattice of partitions of ω

By (ω) we denote the set of all partitions of ω. Let $(\omega)^\omega$ denote the set of all infinite partitions of ω, and let $(\omega)^{<\omega}$ denote the set of all finite partitions of ω. If $X, Y \in (\omega)$, we write $X \leqslant Y$ if X is coarser than Y, that is, each piece of X is a union of pieces of Y. The set of all minimal members of pieces of X is denoted leaders(X). The natural order of leaders(X) induces an order of X. We write $X \leqslant_n Y$ for some $0 < n < \omega$, if, from the nth piece on, every piece of X is a union of pieces of Y. We write $X \leqslant^* Y$ if $X \leqslant_n Y$ for some $0 < n < \omega$. Finally, for $n < \omega$ by $X \restriction n$ we denote the partition of n induced by X. Then $((\omega), \leqslant)$ is a lattice, with least element $\mathbf{0} = \{\omega\}$, and greatest element $\mathbf{1} = \{\{n\} : n < \omega\}$. Given $X, Y \in (\omega)$ and $a, b \in X$, we say that a and

b are linked by Y, if there exist finitely many $a_0, \ldots, a_n \in X$ and $c_0, \ldots, c_{n-1} \in Y$ such that $a_0 = a$, $a_n = b$, and $a_i \cap c_i \neq \emptyset$ and $c_i \cap a_{i+1} \neq \emptyset$ for all $i < n$. Note that there is one piece of $X \wedge Y$ containing a and b iff a and b are linked by Y. By Fr we denote the collection of all $X \in (\omega)$ such that all pieces of X are finite and almost all pieces of X are singletons.

The familiar cardinal invariants can be naturally defined for (ω). A partition of (ω) is a set $\mathscr{A} \subseteq (\omega) \setminus \{\mathbf{0}\}$ such that any two members have meet $\mathbf{0}$, and $\mathscr{A}$ is maximal with this property. Let $\mathfrak{a}((\omega))$ be the minimal cardinality $\geqslant 2$ of a partition of (ω). Accordingly, a partition of $(\omega)^\omega$ is a set $\mathscr{A} \subseteq (\omega)^\omega$ such that any two members have meet in $(\omega)^{<\omega}$, and $\mathscr{A}$ is maximal like this. Then $\mathfrak{a}((\omega)^\omega)$ is the minimal cardinality $\geqslant 2$ of a partition of $(\omega)^\omega$. In [14], Matet has shown that $\mathfrak{p} \leqslant \mathfrak{a}((\omega))$. The same proof gives $\mathfrak{p} \leqslant \mathfrak{a}((\omega)^\omega)$. Krawczyk (see [6]) has shown that $\mathfrak{a}((\omega)^\omega) = \mathfrak{c}$. Halbeisen [9] and Krawczyk asked whether $\mathfrak{a}((\omega)) = \mathfrak{c}$. Below we give a positive answer. We also investigate the distributivity number $\mathfrak{h}((\omega)^\omega)$ of $(\omega)^\omega$. Given $\mathscr{A}, \mathscr{B}$ partitions of $(\omega)^\omega$, we say that $\mathscr{A}$ refines $\mathscr{B}$ if for every $X \in \mathscr{A}$ there is $Y \in \mathscr{B}$ such that $X \leqslant^* Y$. Then $\mathfrak{h}((\omega)^\omega)$ is defined as the minimal size of a family of partitions of $(\omega)^\omega$ such that there is no partition refining all of them. It is easy to see that $\mathfrak{h}((\omega)^\omega) \leqslant \mathfrak{h}$ (see [7]), where $\mathfrak{h}$ is the distributivity number of $\mathscr{P}(\omega)/fin$. Halbeisen and others asked whether $\mathfrak{h}((\omega)^\omega) < \mathfrak{h}$ is consistent. We show that this holds in the Mathias model. The proof is inspired by Brendle's proofs to distinguish between covering coefficients of different tree ideals (see [5]). We shall prove almost every detail, because the referee asked for it.

Theorem 4.1. $\mathfrak{a}((\omega)) = \mathfrak{c}$.

Proof. Suppose $\mathscr{A}$ were a partition of (ω) such that $2 \leqslant |\mathscr{A}| < \mathfrak{c}$. Choose $N \prec H(|\mathscr{A}|^+)$ such that $\mathscr{A} \cup \{\mathscr{A}\} \subseteq N$ and $|N| = |\mathscr{A}|$.

First we assume that $\mathscr{A} \cap Fr \neq \emptyset$. Then clearly $\mathscr{A}$ contains only one element of Fr, say X. Let $a = \{n : \{n\} \in X\}, b = \omega \setminus a$. Then b is finite, and since $|\mathscr{A}| \geqslant 2$ we have that $|b| \geqslant 2$. Note that for every $Z \in \mathscr{A} \setminus \{X\}$ and $c \in Z$, $c \cap b \neq \emptyset$. Therefore $|Z| \leqslant |b|$. Choose $x \in [a]^\omega \setminus N$, and let $i \in b$. Define

$$Y = \{x \cup \{i\}, \omega \setminus (x \cup \{i\})\}.$$

Clearly $X \wedge Y = \mathbf{0}$ since the piece of X which contains i meets $b \setminus \{i\}$. Let $Z \in \mathscr{A} \setminus \{X\}$. By the choice of x and since Z is finite, there must exist $c \in Z$ with $c \cap x \neq \emptyset$ and $c \cap (a \setminus x) \neq \emptyset$. We conclude $Y \wedge Z = \mathbf{0}$. We have shown that $\mathscr{A}$ is not maximal, a contradiction.

Second, we have the case that $\mathscr{A}$ contains no member of Fr. We may identify ω with ${}^{<\omega}\omega$ and hence assume that every member of $\mathscr{A}$ is a partition of ${}^{<\omega}\omega$. We claim that this identification can be done in such a way that for every $X \in \mathscr{A}$ and $y \in {}^\omega\omega$ there exist infinitely many n such that $\{y \restriction n\} \notin X$. In order to see this, for $X \in \mathscr{A}$ let $a_X = \{n < \omega : \{n\} \in X\}$. By case assumption we have that every a_X is coinfinite. Moreover, since no two partitions in $\mathscr{A}$ have a piece in common, we have that the a_X are pairwise disjoint, and hence $a_X \neq \emptyset$ for only countably many X. Let $(X_n : n < \omega)$ be a list all $X \in \mathscr{A}$ with $a_X \neq \emptyset$. Now we identify ω with ${}^{<\omega}\omega$ in such a way that X_n is identified with a subset of ${}^{n+1}\omega$. Then $\mathscr{A}$ will have the desired property.

Fix $x \in {}^{\omega}\omega \setminus N$. Let $a = \{x \restriction n : n < \omega\}$ and $b = {}^{<\omega}\omega \setminus a$. Note that every $c \in [{}^{<\omega}\omega]^{\omega} \cap N$ has infinite intersection with b, as otherwise x is definable in N as the only infinite branch of the tree determined by c. Let $(X_n : n < \omega)$ list all members X of $\mathscr{A}$ such that $X \cap [a]^{<\omega}$ has infinitely many members of size at least 2. Let $(Z_n : n < \omega)$ list all infinite sets $Z \subseteq a$ such that $\{\{\sigma\} : \sigma \in Z\} = X \cap \{\{x \restriction n\} : n < \omega\}$ for some $X \in \mathscr{A}$. Note that by construction $a \setminus Z_n$ is infinite for every n. It is now easy to build $\{a_n : n < \omega\}$ an infinite partition of a such that

(1) for every $n, k, l < \omega$, X_n links a_k and a_l,

(2) for every $n, m < \omega$, $a_n \cap Z_m$ and $a_n \setminus Z_m$ are both infinite.

Define

$$Y = \{a_0 \cup b\} \cup \{a_n : 0 < n < \omega\}.$$

Claim. *$X \wedge Y = \mathbf{0}$, for all $X \in \mathscr{A}$.*

Proof of the Claim. Let $X \in \mathscr{A}$ be arbitrary and let $\mathscr{F} = X \cap [a]^{<\omega}$. We distinguish three cases.

Case 1: $\mathscr{F}$ is finite. Let $n < \omega$ be arbitrary. Since a_n is infinite there exists $c \in X$ such that $c \cap (a_n \setminus \bigcup \mathscr{F}) \neq \emptyset$. If c is finite we have $c \cap b = \emptyset$ by definition of $\mathscr{F}$. If c is infinite, $c \cap b \neq \emptyset$ by the remark after the definition of b above. We conclude that any two pieces of Y are linked by X, and so $X \wedge Y = \mathbf{0}$.

Case 2: $\mathscr{F}$ has infinitely many members of size at least 2. Then $X = X_n$ for some n, and by construction any two pieces of Y are linked by X.

Case 3: $\mathscr{F}$ is infinite and $\mathscr{F} \subseteq^* [a]^1$. Then $Z_n = \{x \restriction k : \{x \restriction k\} \in X\}$ for some n. Let $w = a \setminus \bigcup \mathscr{F}$. By construction and the specific choice of the identification of ω and ${}^{<\omega}\omega$ above, w is infinite, and $a_m \cap w$ is infinite for all $m < \omega$. Moreover, for every $c \in X$ with $c \cap w \neq \emptyset$ we have $c \cap b \neq \emptyset$. Let $0 < n < \omega$. By the above remarks there exists $c \in X$ such that $c \cap a_n \cap w \neq \emptyset$. Hence, we have $c \cap b \neq \emptyset$, and therefore X links a_n and $a_0 \cup b$. We conclude that $X \wedge Y = \mathbf{0}$. □

Theorem 4.2. *Suppose $V \models ZFC + GCH$ and $(P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leqslant \omega_2, \beta < \omega_2)$ is a countable support iteration of Mathias forcing and G is P_{ω_2}-generic over V. Then $V[G]$ is a model for $\mathfrak{h}((\omega)^\omega) = \omega_1$ and $\mathfrak{h} = \omega_2$.*

Proof. The fact that $V[G]$ satisfies $\mathfrak{h} = \omega_2$ is well-known. For a proof see [19]. To prove $\mathfrak{h}((\omega)^\omega) = \omega_1$ in $V[G]$ we will make use of the following simple lemma.

Lemma 4.3. *Let $\{X_n : n < \omega\} \subseteq (\omega) \setminus \{\mathbf{0}\}$ and $Y \in (\omega)^\omega$. There exists $Z \in (\omega)^\omega$ such that $Z \leqslant Y$ and for every $n < \omega$, no piece of X_n is a union of pieces of Z.*

Proof of Lemma 4.3. Let $(c_n : n < \omega)$ list all subsets of ω which occur as a piece of some X_m. Inductively we construct partial partitions $(y_n(i) : i < n)$, for every $n < \omega$, such that $y_n(i)$ is the union of finitely many pieces of Y and $(y_{n+1}(i): i<n+1)$ cannot be extended to a total partition of ω such that c_n is the union of some of its pieces, or equivalently, some $y_{n+1}(i)$ meets c_n and its complement. Suppose we have constructed $(y_n(i): i<n)$.

First, if c_n is not a union of pieces of Y let b be the first piece of Y which meets c_n and its complement. If b is not contained in $y_n(i)$, $i<n$, let $y_{n+1}(i)=y_n(i)$ for $i<n$, and $y_{n+1}(n)=b$. Otherwise let $y_{n+1}(n)$ be the first piece of Y which has not yet been used.

Second, if c_n is a union of pieces of Y, let $B=\{b\in Y: b\subseteq c_n\}$. If B is infinite let $b_0\in B$, $b_1\in Y$ such that $b_0\cap y_n(i)=\emptyset$, $i<n$, and $b_1\nsubseteq c_n$, hence $b_1\cap c_n=\emptyset$ by case assumption. The choice of b_1 is possible by assumption on the X_n. If $b_1\subseteq y_n(i)$ for some $i<n$ let $y_{n+1}(i)=y_n(i)\cup b_0, y_{n+1}(j)=y_n(j)$ for $j<n$, $j\neq i$, and let $y_{n+1}(n)$ be any piece of Y which has not yet been used. Otherwise let $y_{n+1}(i)=y_n(i)$ for $i<n$, and $y_{n+1}(n)=b_0\cup b_1$. Finally suppose that B is finite. If there is $i<n$ such that $y_n(i)$ contains a member of B, choose $b\in Y\setminus B$ which has not yet been used and define $y_{n+1}(i)=y_n(i)\cup b, y_{n+1}(j)=y_n(j)$ for $j<n$, $j\neq i$, and let $y_{n+1}(n)$ be any piece of Y which has not yet been used. Otherwise, there exists $b\in B$ which has not yet been used. Let $c\in Y\setminus B$ such that c has not yet been used. Set $y_{n+1}(i)=y_n(i)$, $i<n$, and $y_{n+1}(n)=b\cup c$. $\square$

For the proof of Theorem 4.2 we shall construct recursively $(\underset{\sim}{\mathscr{A}}^\alpha_\beta: \alpha\leqslant\omega_2, \beta<\omega_1)$ such that the following hold:

(1) $\Vdash_{P_\alpha}$ "$\underset{\sim}{\mathscr{A}}^\alpha_\beta$ is a partition of $(\omega)^\omega$", and, letting $\underset{\sim}{D}_\beta=\{X\in(\omega)^\omega: \exists Y\in\underset{\sim}{\mathscr{A}}^{\omega_2}_\beta(X\leqslant^* Y)\}$, we have $\Vdash_{P_{\omega_2}} \bigcap_{\beta<\omega_1}\underset{\sim}{D}_\beta=\emptyset$.

(2) $\alpha_0<\alpha_1 \Rightarrow \Vdash_{P_{\alpha_1}} \underset{\sim}{\mathscr{A}}^{\alpha_0}_\beta\subseteq\underset{\sim}{\mathscr{A}}^{\alpha_1}_\beta$.

(3) $\mathrm{cf}(\alpha)\geqslant\omega_1 \Rightarrow \Vdash_{P_\alpha} \underset{\sim}{\mathscr{A}}^\alpha_\beta=\bigcup_{\gamma<\alpha}\underset{\sim}{\mathscr{A}}^\gamma_\beta$.

For $\alpha<\omega_2$ choose $\{\underset{\sim}{X}^\alpha_\gamma: \gamma<\omega_1\}$ such that $\Vdash_{P_\alpha} \{\underset{\sim}{X}^\alpha_\gamma: \gamma<\omega_1\}=(\omega)\setminus\{\mathbf{0}\}$. Define $\underset{\sim}{\mathscr{A}}^{<\alpha}_\beta=\bigcup_{\gamma<\alpha}\underset{\sim}{\mathscr{A}}^\gamma_\beta$.

We have to define $(\underset{\sim}{\mathscr{A}}^\alpha_\beta: \beta<\omega_1)$ for α successor or $\mathrm{cf}(\alpha)=\omega$. This will be done such that for every $\underset{\sim}{X}\in\underset{\sim}{\mathscr{A}}^\alpha_\beta\setminus\underset{\sim}{\mathscr{A}}^{<\alpha}_\beta$ the following are satisfied:

(i) $\Vdash_{P_\alpha}$ "no piece of $\underset{\sim}{X}^\alpha_\gamma$ is a union of pieces of $\underset{\sim}{X}$, for every $\gamma<\beta$".

(ii) $\Vdash_{P_\alpha}$ "$\forall Y\in(\omega)^\omega\forall\alpha'<\alpha(Y\leqslant^*\underset{\sim}{X}\Rightarrow Y\notin V[\underset{\sim}{G}_{\alpha'}])$".

Let G_α be P_α-generic over V and let $(r_\nu: \nu<\alpha)$ be the corresponding sequence of Mathias reals. We work in $V[G_\alpha]$. Let $\mathscr{A}^{<\alpha}_\beta=\underset{\sim}{\mathscr{A}}^{<\alpha}_\beta[G_\alpha]$, $X^\alpha_\gamma=\underset{\sim}{X}^\alpha_\gamma[G_\alpha]$. If $\alpha=\alpha'+1$ let $f=r_{\alpha'}$. If $\mathrm{cf}(\alpha)=\omega$, let $(\alpha_n: n<\omega)$ be cofinal in α, let $f\in{}^\omega\omega$ be increasing such that $f\geqslant^* r_{\alpha_n}$ for all $n<\omega$. Since a Mathias real is dominating, in both cases we have that if $g\in{}^\omega\omega$ is such that for some $k<\omega$, for every $n<\omega$, $g(n+k)\geqslant f(n)$, then $g\notin\bigcup_{\alpha'<\alpha}V[G_{\alpha'}]$.

Using Lemma 4.3, it is easy to extend $\mathscr{A}^{<\alpha}_\beta$ to $\mathscr{A}^\alpha_\beta$ a partition of $(\omega)^\omega$ such that if $X\in\mathscr{A}^\alpha_\beta\setminus\mathscr{A}^{<\alpha}_\beta$ then no piece of X^α_γ is a union of pieces of X, for every $\gamma<\beta$, and moreover, if x is the increasing enumeration of leaders(X), $x(i)\geqslant f(i)$ for all $i\in\omega\setminus\{0\}$.

Note that this implies $Y \notin \bigcup_{\alpha'<\alpha} V[G'_\alpha]$ whenever $Y \in (\omega)^\omega$ and $Y \leqslant^* X$. In V we have a P_α-name $\underset{\sim}{\mathscr{A}}^\alpha_\beta$ for $\mathscr{A}^\alpha_\beta$ such that the above properties of $\mathscr{A}^\alpha_\beta$ are forced to hold for $\underset{\sim}{\mathscr{A}}^\alpha_\beta$. This finishes the construction of $(\underset{\sim}{\mathscr{A}}^\alpha_\beta : \alpha \leqslant \omega_2, \beta < \omega_1)$.

Note that we only have to show $\Vdash_{P_{\omega_2}} \bigcap_{\beta<\omega_1} \underset{\sim}{D}_\beta = \emptyset$. For this the following lemma will be crucial.

Recall that for conditions $(s,a),(t,b)$ of Mathias forcing and $n<\omega$ the ordering $(s,a) \leqslant_n (t,b)$ is defined as follows: $(s,a) \leqslant (t,b)$, $s=t$, and a and b have the same first n elements. For $p,q \in P_{\omega_2}$, $F \in [\omega_2]^{<\omega}$ and $n<\omega$ the ordering $p \leqslant_{F,n} q$ is defined by $p \leqslant q$, and for every $\alpha \in F$, $p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \leqslant_n q(\alpha)$.

Define $h_{n,F} \in {}^\omega\omega$ by letting

$$h_{n,F}(i) = 2^{(i+\max\{n,|F|\})^2}.$$

Lemma 4.4. *Suppose* $p \Vdash_{P_{\omega_2}}$ "$\underset{\sim}{X} \in (\omega)^\omega$", $n<\omega$ *and* $F \in [\omega_2 \setminus \{0\}]^{<\omega}$. *There exists* $q \leqslant_{F\cup\{0\},n} p$, *and* $(S^\sigma_i : \sigma \in [\omega]^{<\omega},\ i < h_{n,F}(|\sigma|))$ *such that* $S^\sigma_i \in (\omega)$ *and for all* $g : [\omega]^{<\omega} \to \omega$ *we have*

$$q \Vdash_{P_{\omega_2}} \exists^\infty \sigma \, \exists i < h_{n,F}(|\sigma|)(\underset{\sim}{X} \restriction g(\sigma) = S^\sigma_i \restriction g(\sigma)).$$

Proof of Lemma 4.4. Recall the following fact, which, except for a misprint, is [2, Lemma 9.5].

Lemma 4.5. *Suppose* $x \in V$ *is finite and* $p \Vdash_{P_\alpha} \underset{\sim}{a} \in x$ *(where* $\alpha \leqslant \omega_2$*). For any finite* $F \subseteq \alpha$ *and any* $n<\omega$, *there is* $q \leqslant_{F,n} p$ *and* $y \subseteq x$ *such that* $|y| \leqslant 2^{n\cdot|F|}$ *and* $q \Vdash_{P_\alpha} \underset{\sim}{a} \in y$.

Combining Lemma 4.5 with a fusion argument we obtain the following, which is a straightforward generalization of [2, Lemma 9.6].

Lemma 4.6. *Suppose* $(x_i : i<\omega) \in V$ *is a sequence of finite sets. If* $p \Vdash_{P_\alpha}$ "$\forall i(\underset{\sim}{f}(i) \in x_i)$", *for any finite* $F \subseteq \alpha$ *and* $n<\omega$ *there exist* $q \leqslant_{F,n} p$ *and a sequence* $(y_i : i<\omega) \in V$ *such that for all* i, $y_i \subseteq x_i$ *and* $|y_i| \leqslant h_{n,F}(i)$, *and* $q \Vdash_{P_\alpha} \forall i(\underset{\sim}{f}(i) \in y_i)$.

We are ready to start with the proof of Lemma 4.4. Let $r \in [\omega]^\omega$ be Mathias-generic over V, containing $p(0)$ in its associated generic filter, that is, if $p(0)=(s,a)$ then $s \subseteq r \subseteq s \cup a$. We will confuse r with its increasing enumeration in ${}^\omega\omega$. In $V[r]$, $P_{\omega_2}/\underset{\sim}{G}_0$ is equivalent to P_{ω_2} defined in $V[r]$, by [2, Section 5]. Let x_i be the set of all partitions of $r(i)$. Hence x_i is finite and $(x_i : i<\omega) \in V[r]$. Define a $P_{\omega_2}/\underset{\sim}{G}_0$-name $\underset{\sim}{f}$ for a function in $\prod_{i<\omega} x_i$ by $\underset{\sim}{f}(i) = \underset{\sim}{X} \restriction r(i)$. Applying Lemma 4.6 in $V[r]$ we obtain $q_1 \leqslant_{F,n} p \restriction [1,\omega_2)[r]$ and $y=(y_i : i<\omega)$ as there. We may certainly assume that y_i is a set of $h_{n,F}(i)$ partitions of $r(i)$. As r was arbitrary, in V we can find Q_0-names $\underset{\sim}{q}_1$ and $\underset{\sim}{y} = (\underset{\sim}{y}_i : i<\omega)$, such that $p(0)$ forces the above properties of q_1 and y to hold for $\underset{\sim}{q}_1$ and $\underset{\sim}{y}$.

Let $p(0)=(s,a)$, and let l^* be the nth element of a. We may assume that $s \neq \emptyset$.

Claim 1. *There exists $(s,b) \leqslant_n (s,a)$ such that for every $t \in [b]^{<\omega}$ there exists $\{R_j^{s\cup t} : j < h_{n,F}(|s\cup t|-1)\}$, a set of partitions of $\max(s\cup t)$, such that*

$$(s\cup t, b\setminus \max\{\max(t)+1, l^*+1\}) \Vdash_{Q_0} \underset{\sim}{y}_{|s\cup t|-1} = \{R_j^{s\cup t} : j < h_{n,F}(|s\cup t|-1)\}.$$

Proof of Claim 1. This is a combination of a fusion argument and the pure decision property of Mathias forcing by which we mean the statement of Lemma 4.5 with $\alpha=1$, $n=0$, $F=\{0\}$. Inductively we construct $(b_l : l<\omega)$ and $(k_l : l<\omega)$ such that
(1) $b_0=a$, $k_0=l^*$, $b_{l+1}\in[b_l]^\omega$, $k_{l+1}=\min(b_{l+1})$, $k_{l+1}>k_l$,
(2) for every nonempty $u \subseteq a\cap k_l+1$, (u, b_{l+1}) decides the value of $\underset{\sim}{y}_{|u|-1}$.
Suppose we have got b_l and k_l. By Lemma 4.5, we can shrink b_l 2^{k_l+1} times, thereby getting b_{l+1}, such that (2) holds. For this, note that every nonempty $u \subseteq k_l+1$ determines $r \restriction |u|$, and hence $(u, b_l\setminus(k_l+1))$ forces that there are finitely many possibilities for $\underset{\sim}{y}_{|u|-1}$. So by the pure decision property, we can decide $\underset{\sim}{y}_{|u|-1}$ by only shrinking the second coordinate, and we list the elements as in the Claim. Finally let $b=(a\cap(l^*+1))\cup\{k_l : l<\omega\}$. This finishes the proof of Claim 1. $\square$

Claim 2. *There exist $(s,c)\leqslant_n(s,b)$ and $(S_j^{s\cup t} : t\in[c]^{<\omega}, j<h_{n,F}(|s\cup t|))$, such that $S_j^{s\cup t}\in(\omega)$ and for every $t\in[c]^{<\omega}$ and $k\in\omega$ there exists $m<\omega$ such that $S_j^{s\cup t}\restriction k = R_j^{s\cup t\cup\{i\}}\restriction k$ for all $i\in c\setminus m$ and $j<h_{n,F}(|s\cup t|)$.*

Proof of Claim 2. The proof consists of the combination of a fusion and a compactness argument. We construct $(c_l : l<\omega)$, $(k_l : l<\omega)$ and $(S_j^{s\cup t} : t\in[b]^{<\omega}, j<h_{n,F}(|s\cup t|))$, such that
(1) $c_0=b$, $k_0=l^*$, $c_{l+1}\in[c_l]^\omega$, $k_{l+1}=\min(c_{l+1})$, $k_{l+1}>k_l$,
(2) for every $t\subseteq k_l+1\cap b$, for every $k<\omega$ there exists $m<\omega$ such that $S_j^{s\cup t}\restriction k = R_j^{s\cup t\cup\{i\}}\restriction k$ for all $i\in c_l\setminus m$ and $j<h_{n,F}(|s\cup t|)$.
Suppose we have got c_l and k_l. Fix $t\subseteq b\cap k_l+1$. Consider

$$((R_j^{s\cup t\cup\{i\}} : j<h_{n,F}(|s\cup t|)) : i\in c_l). \tag{$*$}$$

This is a subfamily of the set T of all $h_{n,F}(|s\cup t|)$-tuples of partitions of i, for some $i<\omega$. The set T carries a natural tree structure, defined by coordinatewise extension of partitions, where a partition R of i extends a partition S of j iff $j\leqslant i$ and $R\restriction j=S$. Clearly T is a finitely branching tree of height ω, where the ith level, $i>0$, is the set of all partitions of $i-1$. Hence, by König's Lemma, the subtree generated by the family $(*)$ has an infinite branch, i.e., there exist $(S_j^{s\cup t} : j<h_{n,F}(|s\cup t|))$ and $c'\in[c_l]^\omega$, such that $S_j^{s\cup t}\in(\omega)$ and for every $k<\omega$ there is $m<\omega$ such that $S_j^{s\cup t}\restriction k = R_j^{s\cup t\cup\{i\}}\restriction k$ for all $i\in c'\setminus m$ and $j<h_{n,F}(|s\cup t|)$.

We continue shrinking c', taking care of all $t\subseteq b\cap k_l+1$, and thus after finitely many steps obtain $c_{l+1}\in[c_l]^\omega$ as desired.

Finally let $c=(b\cap l^*+1)\cup\{k_l : l<\omega\}$. This finishes the proof of Claim 2. $\square$

We continue with the proof of Lemma 4.4. Let $q=(s,c)*\underset{\sim}{q}_1$. For σ not of the form $s\cup t$ for some $t\in[c]^{<\omega}$, S_i^σ can be defined arbitrarily. In V, let $g:[\omega]^{<\omega}\to\omega$ and $q'\leqslant q$ be arbitrary. Let $q'(0)=(s\cup t,c')$. By a similar fusion argument as we already gave twice (in the proofs of Claims 1 and 2), we construct $c''\in[c'\backslash l^*+1]^\omega$ such that for every $u\in[c'']^{<\omega}$, $k\in c''\backslash(\max(u)+1)$ and $j<h_{n,F}(|s\cup t\cup u|)$ we have $S_j^{s\cup t\cup u}\restriction g(s\cup t\cup u)=R_j^{s\cup t\cup u\cup\{k\}}\restriction g(s\cup t\cup u)$. So in particular, in this case we will have $k\geqslant g(s\cup t\cup u)$. We claim that $(s\cup t,c'')^\frown q'\restriction[1,\omega_2)\Vdash_{P_{\omega_2}}\exists^\infty\sigma\,\exists j<h_{n,F}(|\sigma|)(\underset{\sim}{X}\restriction g(\sigma)=S_j^\sigma\restriction g(\sigma))$. For the proof, let $r\in[\omega]^\omega$ be Q_0-generic over V, containing $(s\cup t,c'')$ in its associated generic filter. We claim that for every nonempty initial segment σ of r which extends $s\cup t$, in $V[r]$ we have that $q'\restriction[1,\omega_2)[r]\Vdash_{P_{\omega_2}/\underset{\sim}{G}_0}\exists j<h_{n,F}(|\sigma|)(\underset{\sim}{X}\restriction g(\sigma)=S_j^\sigma\restriction g(\sigma))$. This will clearly suffice. So fix such σ, and let $k=\min(r\backslash\sigma)$. By construction of c'' we have that $k\geqslant g(\sigma)$ and $R_j^{\sigma\cup\{k\}}\restriction g(\sigma)=S_j^\sigma\restriction g(\sigma)$, for all $j<h_{n,F}(|\sigma|)$. By Claim 1 we have that

$$\underset{\sim}{y}_{|\sigma|}[r]=\{R_j^{\sigma\cup\{k\}} : j<h_{n,F}(|\sigma|)\}.$$

By construction, in $V[r]$ we have

$$\underset{\sim}{q}_1[r]\Vdash_{P_{\omega_2}/\underset{\sim}{G}_0}\underset{\sim}{X}\restriction k\in\underset{\sim}{y}_{|\sigma|}[r].$$

Consequently, as $(s\cup t,c'')^\frown q'\restriction[1,\omega_2)\leqslant q$, in $V[r]$ we have

$$q'\restriction[1,\omega_2)[r]\Vdash_{P_{\omega_2}/\underset{\sim}{G}_0}\underset{\sim}{X}\restriction g(\sigma)\in\{R_j^{\sigma\cup\{k\}}\restriction g(\sigma) : j<h_{n,F}(|\sigma|)\}.$$

Putting everything together we are done. This finishes the proof of Lemma 4.4. $\square$

The following two lemmas will finish the proof of Theorem 4.2. The first one is [2, Lemma 7.3(a)].

Lemma 4.7. *Suppose $\alpha\leqslant\omega_2$. If $p\Vdash_{P_\alpha}$ "$\underset{\sim}{a}\in V$", $F\in[\alpha]^{<\omega}$ and $n<\omega$, then there exist $q\leqslant_{F,n}p$ and some countable $x\in V$, such that $q\Vdash_{P_\alpha}\underset{\sim}{a}\in x$.*

Lemma 4.8. *Suppose $p\Vdash_{P_{\omega_2}}\underset{\sim}{X}\in(\omega)^\omega$. There exist $q\leqslant p$ and $\gamma<\omega_1$ such that $q\Vdash_{P_{\omega_2}}\underset{\sim}{X}\notin\underset{\sim}{D}_\gamma$.*

Proof. Since new reals are only introduced at stages of the iteration of countable cofinality, we may assume that there is $\alpha<\omega_2$, $cf(\alpha)\leqslant\omega$, such that $p\Vdash_{P_{\omega_2}}\underset{\sim}{X}\in V[\underset{\sim}{G}_\alpha]\backslash\bigcup_{\beta<\alpha}V[\underset{\sim}{G}_\beta]$. Hence we may assume that $\underset{\sim}{X}$ is a P_α-name. Note that by (ii), if for some $\beta<\omega_1$, $p\Vdash_{P_{\omega_2}}$ "$\underset{\sim}{X}\in\underset{\sim}{D}_\beta$", then $p\restriction\alpha\Vdash_{P_\alpha}$ "$\underset{\sim}{X}\in\underset{\sim}{D}_\beta$". We may therefore assume that $p\in P_\alpha$.

Recall from [2, p. 36] that $((p_n,F_n):n<\omega)$ is called a *fusion sequence* if $p_{n+1}\leqslant_{F_n,n}p_n$, $F_n\subseteq F_{n+1}$ for all n, and $\bigcup_{n<\omega}F_n=\bigcup_{n<\omega}\operatorname{supp}(p_n)$. Such sequence has an infimum.

Applying Lemmas 4.4 and 4.7 it is straightforward to construct a fusion sequence $((p_n, F_n): n<\omega)$ in P_α such that

(1) $p_0 \leqslant p$, and there exists $\gamma^* < \omega_1$ such that $p_0 \Vdash_{P_\alpha} \underset{\sim}{X} \in \{\underset{\sim}{X}^\alpha_\beta : \beta < \gamma^*\}$,

(2) for every $\beta \in \operatorname{supp}(p_n)$ there exist $m>n$ and hereditarily countable P_β-names $\underset{\sim}{h}_\beta$ and $(\underset{\sim}{S}^\beta_{\sigma,j}: \sigma \in [\omega]^{<\omega},\ j<\underset{\sim}{h}_\beta(\sigma))$, such that

$$p_m \restriction \beta \Vdash_{P_\beta} \text{“}\underset{\sim}{h}_\beta : [\omega]^{<\omega} \to \omega,\ \underset{\sim}{S}^\beta_{\sigma,j} \in (\omega) \text{ and } \forall g : [\omega]^{<\omega} \to \omega:$$

$$p_m \restriction [\beta, \alpha) \Vdash_{P_\alpha/\underset{\sim}{G}_\beta} \text{“}\exists^\infty \sigma \exists j < \underset{\sim}{h}_\beta(\sigma)(\underset{\sim}{X} \restriction g(\sigma) = \underset{\sim}{S}^\beta_{\sigma,j} \restriction g(\sigma))\text{” ”},$$

(3) for every $\beta \in \operatorname{supp}(p_n)$, all the (countably many) coordinates needed to evaluate $\underset{\sim}{h}_\beta$ and $\underset{\sim}{S}^\beta_{\sigma,j}$ belong to $\operatorname{supp}(p_m)$, for some $m>n$; in particular, if $\beta' < \beta$ and $[\beta', \beta) \cap \operatorname{supp}(p_m) = \emptyset$ for all m, then $\underset{\sim}{S}^\beta_{\sigma,j}$ is a $P_{\beta'}$-name,

(4) for every $\beta \in \operatorname{supp}(p_n)$ there exist $\gamma_\beta < \omega_1$ and $m>n$ such that $p_m \restriction \beta \Vdash_{P_\beta}$ “$\{\underset{\sim}{S}^\beta_{\sigma,j}: \sigma \in [\omega]^{<\omega},\ j<\underset{\sim}{h}_\beta(\sigma)\} \subseteq \{\underset{\sim}{X}^\beta_\nu : \nu < \gamma_\beta\} \cup \{\mathbf{0}\}$”.

Let $q \in P_\alpha$ be the infimum of $(p_n : n<\omega)$, and let $\gamma = \operatorname{supp}\{\gamma^*, \gamma_\beta : \beta \in \operatorname{supp}(q)\}$. The following claim will finish the proof of Lemma 4.8.

Claim. $q \Vdash_{P_{\omega_2}} \underset{\sim}{X} \notin \underset{\sim}{D}_\gamma$.

Proof of the Claim. Otherwise there exist $r \leqslant q$, $\beta < \omega_2$, $0<n<\omega$ and a P_β-name $\underset{\sim}{Y}$ such that

(*) $r \Vdash_{P_{\omega_2}}$ “$\underset{\sim}{Y} \in \underset{\sim}{\mathscr{A}}_\gamma{}^\beta \setminus \underset{\sim}{\mathscr{A}}_\gamma{}^{<\beta}$and $\underset{\sim}{X} \leqslant_n \underset{\sim}{Y}$”, and r decides the first $n+1$ leaders of $\underset{\sim}{X}$, say as $x_0, \ldots, x_n$.

Case 1: $\beta \in \operatorname{supp}(q)$. Choose G_β P_β-generic over V with $r \restriction \beta \in G_\beta$. Let $S^\beta_{\sigma,j} = \underset{\sim}{S}^\beta_{\sigma,j}[G_\beta]$, $h_\beta = \underset{\sim}{h}_\beta[G_\beta]$, $Y = \underset{\sim}{Y}[G_\beta]$. By (i) and (4), no piece of $S^\beta_{\sigma,j}$ is a union of pieces of Y, for all σ and j such that $S^\beta_{\sigma,j} \neq \mathbf{0}$. Define $g : [\omega]^{<\omega} \to \omega \setminus (x_n + 1)$ such that for every $i < h_\beta(\sigma)$, if $|S^\beta_{\sigma,i}| \geqslant n+1$ and a is the nth piece of $S^\beta_{\sigma,i}$, there exists $b \in Y$ such that $b \cap g(\sigma) \cap a \neq \emptyset$ and $b \cap (g(\sigma) \setminus a) \neq \emptyset$. By (2) there exist $r' \leqslant r \restriction [\beta, \omega_2)$, σ and $j < h_\beta(\sigma)$ such that $r' \Vdash_{P_{\omega_2}/G_\beta}$ “$\underset{\sim}{X} \restriction g(\sigma) = S^\beta_{\sigma,j} \restriction g(\sigma)$”. Since $g(\sigma) > x_n$ we know that $|S^\beta_{\sigma,j} \restriction g(\sigma)| \geqslant n+1$. By construction we have that r' forces that the nth piece of $\underset{\sim}{X}$ is not a union of pieces of Y. This contradicts (*).

Case 2: $\beta \notin \operatorname{supp}(q)$. In case $\beta > \alpha$, by (ii) we have that it is forced that no $Z \leqslant^* \underset{\sim}{Y}$ belongs to $V[\underset{\sim}{G}_\alpha]$. We get a contradiction since $\underset{\sim}{X}$ is a P_α-name. In the case $\beta = \alpha$ we get a contradiction by (i) and (1). Hence we are left with the case $\beta < \alpha$. Choose $\alpha > \beta' > \beta$ minimal with $\beta' \in \operatorname{supp}(q)$. Work in $V[G_{\beta'}]$, where $G_{\beta'}$ is $P_{\beta'}$-generic over V, containing $r \restriction \beta'$. Let $S^{\beta'}_{\sigma,j} = \underset{\sim}{S}^{\beta'}_{\sigma,j}[G_{\beta'}]$, $h_{\beta'} = \underset{\sim}{h}_{\beta'}[G_{\beta'}]$, $Y = \underset{\sim}{Y}[G_{\beta'}]$. By (3) we have that $S^{\beta'}_{\sigma,i} \in V[G_\beta]$, and hence by (4) and (ii) we conclude that whenever $|S^{\beta'}_{\sigma,i}| \geqslant 2$, it is forced that no piece of $S^{\beta'}_{\sigma,i}$ is a union of pieces of $\underset{\sim}{Y}$. Using this we reach a contradiction as in Case 1. In fact, in $V[G_{\beta'}]$ define $g:[\omega]^{<\omega} \to \omega \setminus x_n + 1$, such that for every $j < h_{\beta'}(\sigma)$, if $|S^{\beta'}_{\sigma,j}| \geqslant n+1$ and a is the nth piece of $S^{\beta'}_{\sigma,j}$, then there exists $b \in Y$ such that $b \cap g(\sigma) \cap a \neq \emptyset$ and $b \cap (g(\sigma) \setminus a) \neq \emptyset$. By (2) there exist $r' \leqslant r \restriction [\beta', \omega_2)$, σ and $j < h'_{\beta}(\sigma)$ such that $r' \Vdash_{P_{\omega_2}/\underset{\sim}{G}_{\beta'}}$ "$\underset{\sim}{X} \restriction g(\sigma) = S^{\beta'}_{\sigma,j} \restriction g(\sigma)$". Since $g(\sigma) > x_n$ we know that $|S^{\beta'}_{\sigma,j} \restriction g(\sigma)| \geqslant n+1$. By construction we have that r' forces that the nth piece of $\underset{\sim}{X}$ is not a union of pieces of Y. This contradicts $(*)$. $\square$

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